The generation of cone like structures by use of their intersections

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Introduction

Aims

A given property of a cone is that a plane parallel to its slant intersects it with a parabolic curve. This article investigates the 3-dimensional equation of a (infinite) cone with a slant length along the z-axis such that any intersection with a plane perpendicular to the x-axis is a parabolic curve (For simplicity, this equation will assume the graph to be symmetrical about the $x-z$ plane). To further this investigation, I will derive formulae such that other types of curves can be generated from this intersection, as opposed to a parabola. Note that this does not (and cannot) guarantee that a cone will be formed, but a 3D structure can be derived nonetheless.

Methodology for derivation

To derive a formula of a cone with the specifications layed out in the 'Aims' section, I will attempt first to find the equation of the curve generated by a plane perpendicular to the x-axis. For some x', this equation will be of the form $z = f(y)$, where x' is constant. We will later generalise x' as a variable x to achieve the full equation of the cone in the form $z = f(x, y)$.

For clarity, I will proceed with the diagram below to illustrate the desired product along the $z-x$ and $z-y$ plane (for visual cohesion, the dashed line represents an arbitrary circular intersection with the cone):

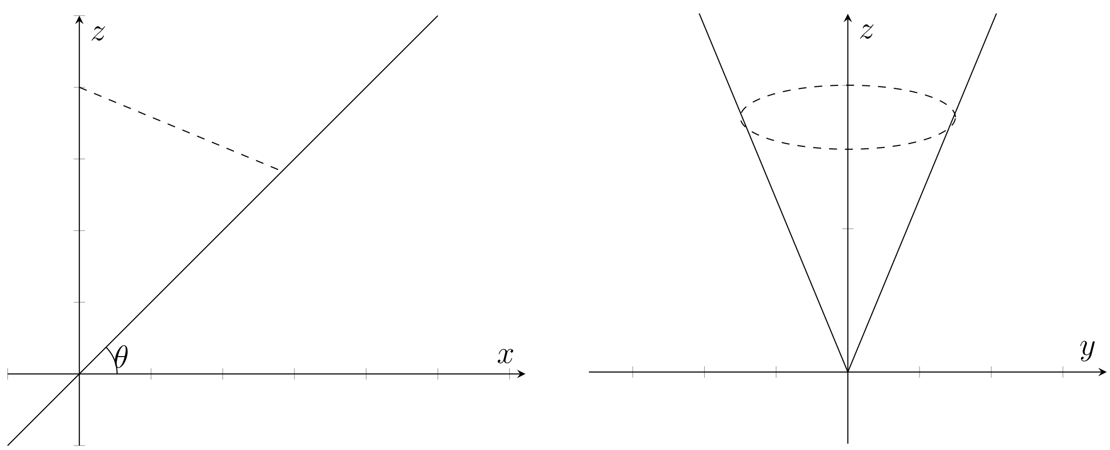


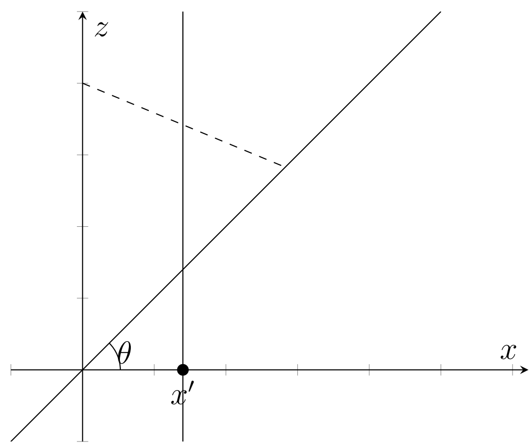
Figure 1: $z-y$ & $z-x$ projections

For generality, I have let θ be an arbitrary angle to generalise the angle of the cone. Additionally, note how a cone will project an isosceles triangle over the $z-x$ plane. In order to produce an equation for a parabola at an arbitrary $x = x'$ intersection, I will use three points (two symmetrical about the $z-x$ plane), and one lying on the bottom slope of the isosceles triangle.

Deriving the equation of the cone

Finding the points of intersection

I will proceed first by identifying three particular points of intersection with the cone along the plane $x = x'$. Because these points are along $x = x'$, I need only find the z and y co-ordinates. To do so, I will use the graphs in figure 1 to find the relevant points. Below is the $x - z$ projection of the cone, with the line projection of the plane $x = x'$:



The first point that can be identified is the intersection of $x = x_1$ and the base of the triangle. Intuitively, as the cone is specified (in 'Aims') to be symmetrical about the $z - x$ plane, this intersection must lie on $y = 0$. The z value, by trigonometry, is $x_1 \tan(\theta)$; I shall define this point as M. The next two points I will derive by creating another intersection with the cone. In figure 2, I showed an arbitrary circular intersection with the cone. I shall do so again, but not arbitrarily. Let *this* intersection be defined such that the plane $x = x'$ bisects it. I will define its projection on the $z - x$ plane to be AB. Furthermore, I shall define its intersection with $x = x'$ to be R. For reference, the points R_1 & R_2 are shown below in the $z - y$ plane, which have the same z co-ordinate as R:

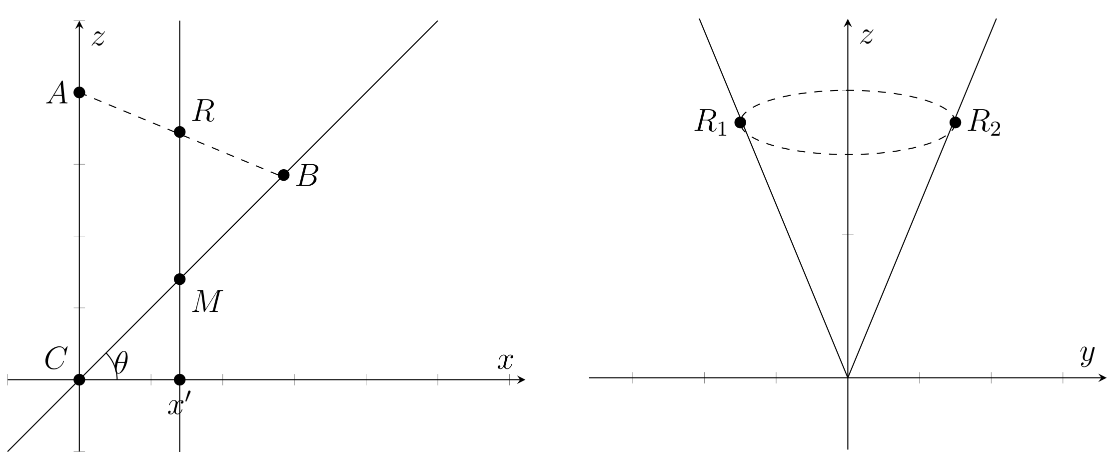
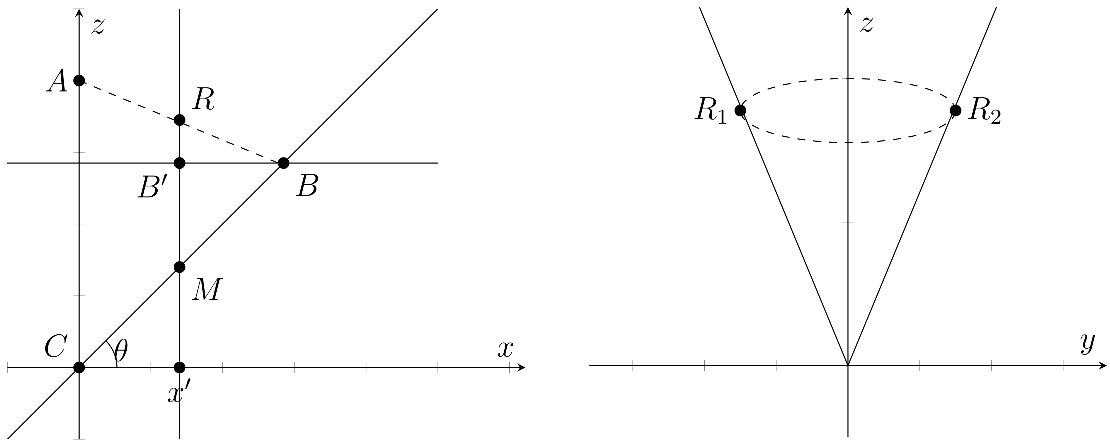


Figure 2: Projections to find R

I can identify the height of R by adding the vertical component of RB' to the height of B. Note that CM is the same length and orientation as MB. Hence the height of B is $2x'\tan(\theta)$. Finding the vertical component of RB is a harder task, but can be achieved by creating a horizontal line about B, as shown below:



From here the geometry is clear. $B\hat{M}R = \frac{\pi}{2} - \theta$, and as MRB is an isosceles triangle, $B\hat{R}B' = \frac{\theta}{2} - \frac{\pi}{4} \Rightarrow R\hat{B}B' = \frac{\pi}{4} - \frac{\theta}{2}$. Hence, $B'R = x'\tan(\frac{\pi}{4} - \frac{\theta}{2})$. Therefore the z co-ordinate of R is $x'\tan(\frac{\pi}{4} - \frac{\theta}{2}) + 2x'\tan(\theta)$.

Note how the circular intersection of the cone is chosen such that R is in the centre. This implies that R_1 & R_2 are the maximum distance from the $z - x$ plane, as seen above. Hence, the magnitude of the y co-ordinate is equivalent to the radius of the circular intersection. We can see from the $z - x$ projection, that this is the distance RB. As a result, the y co-ordinates of R_1 and R_2 is $\pm\frac{x'}{\cos(\frac{\pi}{4}-\frac{\theta}{2})}$.

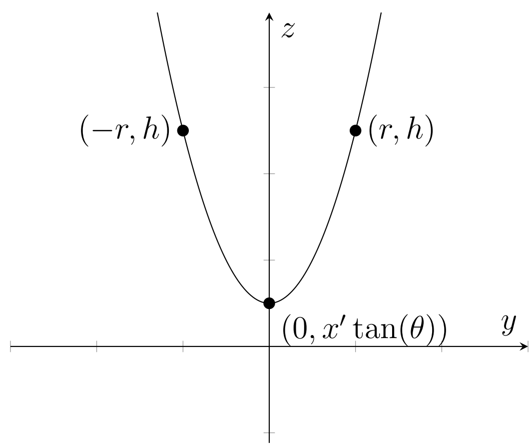
In full, the points are below, along with substitutions in terms of r and h for cohesion in later sections:

- $\left(x', \frac{x'}{\cos(\frac{\pi}{4}-\frac{\theta}{2}}, x'\tan(\frac{\pi}{4} - \frac{\theta}{2}) + 2x'\tan(\theta))\right) \rightarrow (x', r, h)$
- $\left(x', -\frac{x'}{\cos(\frac{\pi}{4}-\frac{\theta}{2}}, x'\tan(\frac{\pi}{4} - \frac{\theta}{2}) + 2x'\tan(\theta))\right) \rightarrow (x', -r, h)$
- $(x', 0, x'tan(\theta))$

Note how each co-ordinate scales linearly with x'; this will be important later.

Defining a parabola and cone from the three points

All three points share the same x co-ordinate. To reiterate, as the slant of the cone also lies on the same x co-ordinate, the intersection of the cone with the plane $x = x'$ must be a parabola. Hence, the three points must lie on a parabolic curve on the $x = x'$ plane. This is shown below:



To find the equation of the parabola that intersects these points I shall write it in the form $z = f(y) = ay^2 + c$. This is the most general equation that can be used, as the cone is specified (in 'aims') to be symmetrical about the $z - x$ plane, and an additional '$+by$' implies a translation parallel to the y-axis. The algebra follows:

$$z = f(y) = ay^2 + x' \tan(\theta)$$
$$f(r) = h = ar^2 + x' \tan(\theta)$$
$$a = \frac{h - x' \tan(\theta)}{r^2}$$
$$\therefore f(y) = x' \tan(\theta) + \frac{h - x' \tan(\theta)}{r^2} y^2$$

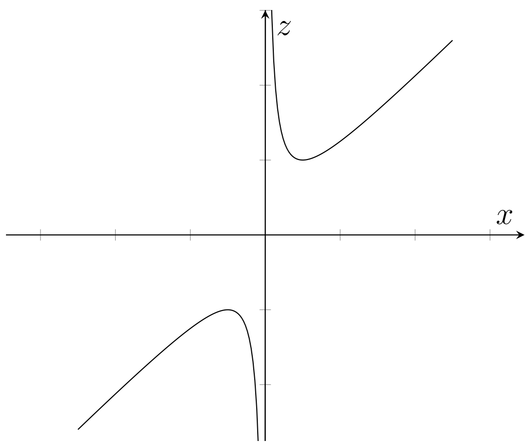
Both a and c are specified, hence this is the only parabola that can be formed from these 3 points. Now the entire cone can be generalised by substituting x' for x. Arbitrarily where $\theta = \frac{\pi}{3}$, the equation below is formed:

$$z = x \tan\left(\frac{\pi}{3}\right) + \left(\frac{\left(2x \tan\left(\frac{\pi}{3}\right) + x \tan\left(\frac{\pi}{4} - \frac{\frac{\pi}{3}}{2}\right)\right) - x \tan \frac{\pi}{3}}{\left(\frac{x}{\cos\left(\frac{\pi}{4} - \frac{\frac{\pi}{3}}{2}\right)}\right)^2} \right) y^2$$

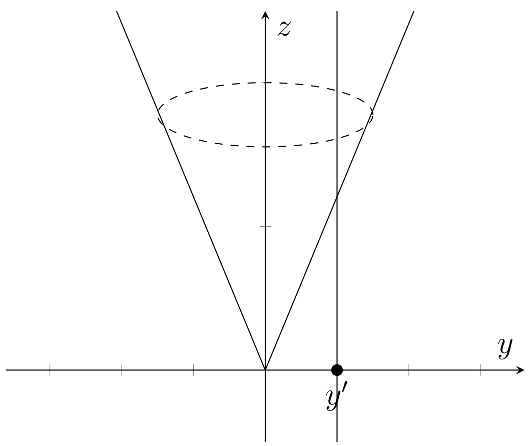
$$= x \tan\left(\frac{\pi}{3}\right) + \left(\frac{\left(x \tan\left(\frac{\pi}{3}\right) + x \tan\left(\frac{\pi}{12}\right)\right)}{\left(\frac{x}{\cos\left(\frac{\pi}{12}\right)}\right)^2} \right) y^2$$

$$= \sqrt{3}x + \frac{\left(2 + \sqrt{3}\right) y^2}{2x}$$

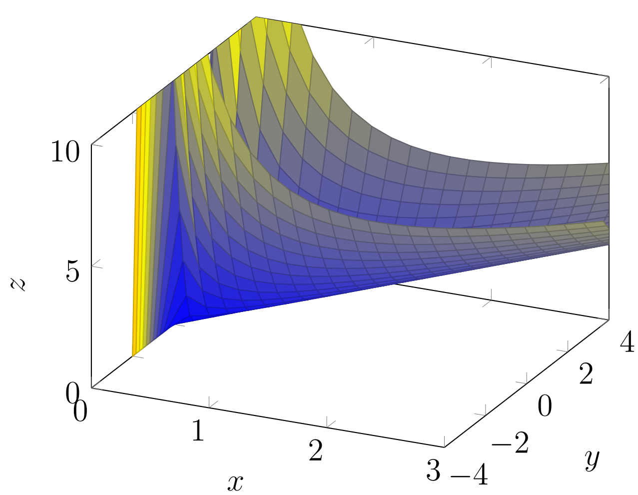
Note how for a constant x, the equation becomes a quadratic in terms of y, as intended. However, for a constant $y = y'$, the equation becomes of the form $ax + \frac{b}{x}$, which (in general) looks like the curve below:



This is a hyperbola, because this is an open intersection of the cone that is not parallel to its slant. This intersection is shown below:



A 3-Dimensional plot of the cone is shown below:



$$z = \sqrt{3}x + \frac{\left(2 + \sqrt{3}\right) y^2}{2x}$$

Deriving other equations from other curves

General cases

Note that it is only because I was aiming for a cone that I used the equation $z = f(y) = ax^2 + c$. However, there are other curves that can be 'forced' through the three points. To select these curves, I will set these restrictions on their equations below:

- It must be an even function
- There can be no asymptotes (This is not strictly necessary, but it allows for more generality later on)
- The equation must intersect $(0, 0)$
- The graph must approach $z = 0$ as x approaches 0. This ensures the 3-D structure is closed on the z-axis

Given these restrictions, I can 'force' the curves through the 3 points by expressing it as $f(y) = ag(y) + c$, where $g(y)$ is a general equation for the curve, and $f(y)$ is the particular case where it intersects the three points. I can now proceed to generalise the method used earlier:

$$\begin{aligned} z = f(y) &= ag(y) + x' \tan(\theta) \\ f(r) = h &= ag(r) + x' \tan(\theta) \\ a &= \frac{h - x' \tan(\theta)}{g(r)} \\ \therefore f(y) &= x' \tan(\theta) + \left(\frac{h - x' \tan(\theta)}{g(r)} \right) g(y) \end{aligned}$$

Note that, unlike a parabola, there is no guarantee that $f(y)$ is unique for each type of curve. As before we have '$h - x' \tan(\theta)$' in the numerator. This permits a small amount of simplification shown below:

$$\begin{aligned} x' \tan(\theta) + \left(\frac{h - x' \tan(\theta)}{g(r)} \right) g(y) &= x' \tan(\theta) + \left(\frac{x' \tan(\frac{\pi}{4} - \frac{\theta}{2}) + 2x' \tan(\theta)) - x' \tan(\theta)}{g(r)} \right) g(y) \\ &= x' \tan(\theta) + \left(\frac{x' \tan(\frac{\pi}{4} - \frac{\theta}{2}) + x' \tan(\theta)}{g(r)} \right) g(y) \end{aligned}$$

Using this formula, I have generated equations for different shapes, each of which come from different expressions for $g(y)$. They are listed below:

$g(y)$	$f(x,y)$
y^4	$x\tan(\theta)+\left(\frac{\left(x\tan(\theta)+x\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}{\left(\frac{x}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)^4}\right)y^4$
$\lim_{n\to\infty}(y^n)$	$\lim_{n\to\infty}\left(x\tan(\theta)+\left(\frac{\left(x\tan(\theta)+x\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}{\left(\frac{x}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)^n}\right)y^n\right)$
$\cosh(y)-1$	$x\tan(\theta)+\left(\frac{\left(x\tan(\theta)+x\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}{\left(\cosh\left(\frac{x}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)\right)-1}\right)(\cosh(y)-1)$
$\sqrt{y^2+1}-1$	$x\tan(\theta)+\left(\frac{\left(x\tan(\theta)+x\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}{\sqrt{\left(\frac{x}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)^2+1}-1}\right)\left(\sqrt{y^2+1}-1\right)$
$\lim_{n\to\infty}\left(\sqrt[n]{y^n+1}-1\right)$	$\lim_{n\to\infty}\left(x\tan(\theta)+\left(\frac{\left(x\tan(\theta)+x\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right)}{\sqrt[n]{\left(\frac{x}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)^n+1}-1}\right)\left(\sqrt[n]{y^n+1}-1\right)\right)$

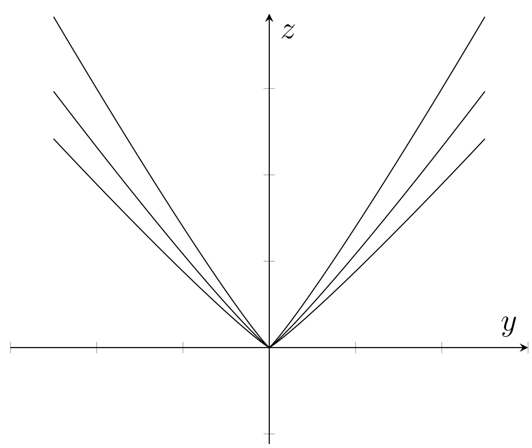
Special cases

In this section, I would like to discuss two equations that do not fit the generalisation $f(y) = ag(y) + c$. The first case is $|x|$. Earlier, I mentioned that each of the three points was a scale of x'. Therefore the linearity of $|x|$ implies that $f(x, y)$ does not vary with x, and hence does not meet the z-axis at x=0.

However, the equation can be adapted by a small degree such that it does. The first thing to note is that $f(x, y) = ax + \frac{bx}{g(r)}g(y) = ax + \frac{bx}{g(cx+d)}g(y)$ where $a, b, c, d \in \mathbb{R}$. Drawing focus on how z looks with respect to y we need only consider $\frac{bx}{g(cx+d)}g(y)$, as ax has no influence on it. Now consider $g(y) = |y|^{1.1}$. This results in the equation below:

$$\frac{bx}{g(cx + d)}g(y) = \frac{bx}{|cx + d|^{1.1}}(|y|^{1.1})$$

Note how $\lim_{x\to\infty}\left(\frac{x}{|cx+d|^{1.1}}\right)$ approaches a constant value. However, for $\lim_{x\to 0}\left(\frac{x}{|cx+d|^{1.1}}\right)$, the value approaches ∞. As a result, $\lim_{n\to\infty}(n|y|^{1.1}$ becomes the equation of a straight line. As the coefficient of $|y|^{1.1}$ becomes larger (from a smaller value of x), the curve like-properties become more apparent:



However, the line 'snapping' to the z-axis happens over a finite region of x. In order to make this instantaneous, effectively defining a plane over an infinitesimally small δx, I used these equation:

$$g(y) = \lim_{n\to 1^+}|y|^n$$

$$f(x, y) = \lim_{n\to 1^+}\left(x\tan(\theta) + \left(\frac{\left(x\tan(\theta) + x\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)}{\left|\left(\frac{x}{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right)\right|^n}\right)|y|^n\right)$$

The second special case is $\sec(x) - 1$. The restriction this does not obey is that it is asymptotal. This is problematic as the generalisation of $f(y)$ relies on transforming $g(x)$ onto the three points by stretching the graph along the z-axis. If the three points lie past, or on the asymptote, the transformation will not work, or yield bizarre results due to the periodicity of the function. To work around this, I have devised a method below, by changing the generalisation of $f(y)$, to stretch the graph parallel to the $y - axis$:

$$\begin{aligned} z = f(y) &= \sec(my) - 1 + x' \tan(\theta) \\ f(r) = h &= \sec(mr) - 1 + x' \tan(\theta) \\ \therefore m &= \frac{\operatorname{arcsec}\left(h + 1 - x' \tan(\theta)\right)}{r} \\ \therefore f(y) &= \sec\left(\frac{\operatorname{arcsec}\left(h + 1 - x' \tan(\theta)\right)}{r} y\right) - 1 + x' \tan(\theta) \end{aligned}$$

Note that I did not need to use this method exclusively for the secant function; it is entirely possible that some curves prior could have drastically different equations if I applied this method. Either way, the final result for $g(y) = \sec(y) - 1$ is below:

$$\begin{aligned} f(x, y) &= x \tan(a) + \sec\left(\frac{\operatorname{arcsec}\left(2x \tan(\theta) + x \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + 1 - x \tan(\theta)\right)}{\frac{x}{\cos\left(\frac{\pi}{4} - \frac{a}{2}\right)}} y\right) - 1 \\ &= x \tan(a) + \sec\left(\frac{\operatorname{arcsec}\left(2x \tan(\theta) + x \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + 1\right)}{\frac{x}{\cos\left(\frac{\pi}{4} - \frac{a}{2}\right)}} y\right) - 1 \end{aligned}$$